## MLR Inference II: *Inference and Assessment Metrics Converge*

- ***t Stats and Incremental Goodness-of-Fit***
- ***… and WhatsNew about x:***
- ***Comparing MLR Models II: t stats and adjusted R²***
- ***… Proof (Appendix)***

### *t Stats and Incremental Goodness-of-Fit*

1. In SLR Inference, you saw the convergence of inference and assessment metrics, driven by relationship between t statistics and the $R^2$ measure of goodness of fit, as well as SSE/SSR:

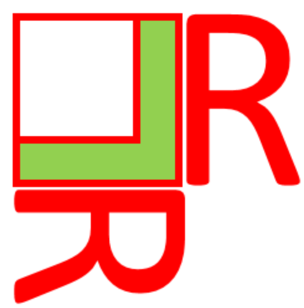
$$t_{\hat{\beta}_1}^2 = (n-2)\frac{R^2}{1-R^2} = (n-2)\frac{SSE}{SSR}.$$

2. Those relationships highlighted the fact that precision in estimation is jointly driven by sample size and Goodness-of-Fit, and that large samples sizes or high $R^2$ alone would not individually assure precision in estimation.

3. It turns out that we have similar results in MLR models. Precision of estimation is jointly driven by the degrees of freedom (*dofs*) and now the marginal or incremental impact that each RHS variable has on $R^2$ or $SSE$'s:[1]

$$t_{\hat{\beta}_x}^2 = dofs\frac{\Delta R_x^2}{1-R^2} = dofs\frac{\Delta SSE_x}{SSR}$$

where $dofs = n-k-1$, and $\Delta R_x^2$ $(\Delta SSE_x)$ is the increase in $R^2$ $(SSE)$ when *x* is the *last* variable added to the model.



4. This equation makes clear what we previously saw with SLR models:

5. The SLR and MLR formulas are in fact consistent here, once you realize that $R^2$ in an SLR model is in fact the same as $\Delta R_x^2$ when going from no RHS variables (other than the constant term) to having the one RHS variable *x* in the SLR model. Or put differently, $\Delta R_x^2 = R^2 - 0 = R^2$ is the increase in $R^2$ when x is introduced to the SLR model, and likewise for $SSE$. So the SLR and MLR formulas are in fact consistent, and in both cases, t stat magnitudes reflect *dofs* as well as the incremental $R^2$ $(SSE)$ when variables are added to the model.

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[1] The proof of this relationship will come later when we explore the relationship between t stats and F statistics.

6. **Example**: Here's an example, working with the bodyfat dataset, and a Full Model with *hgt*, *wgt* and *abd* on the RHS. To determine the marginal impact each RHS variable has on RHS, we first estimate three models dropping one explanatory variable in each (Models (1)-(3)), and then the Full Model (Model (4):

| | Dropping One RHS Variable | | | Full Model |
|---|---|---|---|---|
| | (1) | (2) | (3) | (4) |
| | brozek | brozek | brozek | brozek |
| wgt | 0.187*** | -0.136*** | dropped | -0.120*** |
| | (14.48) | (-7.08) | | (-5.41) |
| hgt | -0.650*** | dropped | -0.342*** | -0.118 |
| | (-6.29) | | (-4.55) | (-1.43) |
| abd | dropped | 0.915*** | 0.595*** | 0.880*** |
| | | (17.42) | (23.30) | (15.19) |
| _cons | 31.16*** | -41.35*** | -12.12* | -32.66*** |
| | (4.51) | (-17.14) | (-2.17) | (-5.01) |
| N | 252 | 252 | 252 | 252 |
| R-sq | 0.4614 | 0.7187 | 0.6881 | 0.7210 |
| mss (SSE) | 6,958.1 | 10,837.7 | 10,375.8 | 1,0872.6 |
| rss (SSR) | 8,121.0 | 4,241.3 | 4,703.2 | 4,206.5 |

t statistics in parentheses
* p<0.05, ** p<0.01, *** p<0.001

Looking at *abd* as the *last* variable, so comparing Models (1) and (4):

$$t^2_{\hat{\beta}_{abd}} = (dofs)\frac{\Delta R^2_{abd}}{1-R^2} = 248\frac{.7210-.4614}{1-.721} = 248\frac{.2596}{1-.721} = (15.19)^2$$

$$t^2_{\hat{\beta}_{abd}} = dofs\frac{\Delta SSE_{abd}}{SSR} = 248\frac{10{,}872.6-6{,}958.1}{4{,}206.5} = 248\frac{3{,}914.5}{4{,}206.5} = (15.19)^2$$

And so as advertised, $t^2_{\hat{\beta}_x} = dofs\frac{\Delta R^2_x}{1-R^2} = dofs\frac{\Delta SSE_x}{SSR}$ .

7. Notice also that in looking across the various t stats in an MLR model, you see that the square of the t stats, $t^2_{\hat{\beta}_x}$ , are directly proportional to each variable's marginal/incremental contribution to $R^2$ and to $SSE$'s: $\frac{t^2_{\hat{\beta}_x}}{t^2_{\hat{\beta}_z}} = \frac{\Delta R^2_x}{\Delta R^2_z} = \frac{\Delta SSE_x}{\Delta SSE_z}$, for any two RHS variables *x* and *z*.

   a. Comparing *wgt* and *abd*:

      i. Since $\Delta R^2_{abd} = .2596$ and $\Delta R^2_{wgt} = .7210-.6881 = .0329$, we have:

$$\frac{\Delta R^2_{abd}}{\Delta R^2_{wgt}} = \frac{.2596}{.0329} = 7.88 = \frac{t^2_{\hat{\beta}_{abd}}}{t^2_{\hat{\beta}_{wgt}}} = \left(\frac{15.19}{5.41}\right)^2$$

ii. And since $\Delta SSE_{abd} = 3{,}914.5$ and $\Delta SSE_{wgt} = 10{,}872.6 - 10{,}375.8 = 496.8$, we have:

$$\frac{\Delta SSE_{abd}}{\Delta SSE_{wgt}} = \frac{3{,}914.5}{496.8} = 7.88 = \frac{t^2_{\hat{\beta}_{abd}}}{t^2_{\hat{\beta}_{wgt}}} = \left(\frac{15.19}{5.41}\right)^2$$

8. So variables with larger t stats have greater marginal impacts on $R^2$ and $SSE$ … and *vice-versa*. ***Who saw this coming?***

***… and WhatsNew about x:***

9. Perhaps not surprisingly, you can find $\Delta R^2_x$ and $\Delta SSE_x$ in the regression of *y* on *WhatsNew* about *x*, where $\Delta R^2_x$ is the $R^2$ in the *WhatsNew* SLR regression, and $\Delta SSE_x$ is the $SSE$ in that model.

10. To see this, let's turn to the previous example, and focus again on the *abd* variable. From above, we know that $\Delta R^2_{abd} = .2596$ and $\Delta SSE_{abd} = 3{,}914.5$. Here are the results from the regression of *brozek* on *WhatsNew* about *abd*, and the results are as advertised:


```
. reg abd wgt hgt
. predict whatsnew, resid

. reg brozek whatsnew

      Source |       SS           df       MS      Number of obs   =       252
-------------+----------------------------------   F(1, 250)       =     87.65
       Model |   3914.4903         1   3914.4903   Prob > F        =    0.0000
    Residual |  11164.5263       250  44.6581053   R-squared       =    0.2596
-------------+----------------------------------   Adj R-squared   =    0.2566
       Total |  15079.0166       251  60.0757635   Root MSE        =    6.6827

------------------------------------------------------------------------------
      brozek |      Coef.   Std. Err.      t    P>|t|     [95% Conf. Interval]
-------------+----------------------------------------------------------------
   whatsnew |    .879846   .0939765     9.36   0.000     .6947594    1.064932
       _cons |   18.93849   .4209688    44.99   0.000     18.10939    19.76759
------------------------------------------------------------------------------
```

## Comparing MLR Models II: t stats *and adjusted $R^2$*

11. It turns out that there's a direct relationship between t stats and changes in adjusted R-sq: with the addition of RHS variables, the movement of $\bar{R}^2$ is directly tied to whether or not the t stats of the added variables are larger than 1 in magnitude, or not.

12. $\bar{R}^2$ will always increase (decrease) when variables with t stats larger (smaller) than one in magnitude are added to the MLR model… and *vice-versa* when dropping variables from a model.

    a. With the addition of a RHS variable: $\bar{R}^2 \begin{bmatrix} increases \\ stays\ the\ same \\ decreases \end{bmatrix} \ when\ |t| \begin{bmatrix} > \\ = \\ < \end{bmatrix} 1.$

    b. This results follows directly from $t_{\hat{\beta}_x}^2 = dofs \frac{\Delta R_x^2}{1-R^2}$ and is proved in the Appendix

13. Here's an example, working with the bodyfat dataset:

| | (1) Brozek | (2) Brozek | (3) Brozek | (4) Brozek |
|---|---|---|---|---|
| hgt | -0.650*** | -0.118 | -0.131 | -0.138 |
| | (-6.29) | (-1.43) | (-1.51) | (-1.55) |
| wgt | 0.187*** | -0.120*** | -0.108** | -0.100* |
| | (14.48) | (-5.41) | (-3.18) | (-2.52) |
| abd | | 0.880*** | 0.883*** | 0.898*** |
| | | (15.19) | (15.13) | (12.62) |
| hip | | | -0.0564 | -0.0723 |
| | | | (-0.49) | (-0.58) |
| chest | | | | -0.0348 |
| | | | | (-0.38) |
| _cons | 31.16*** | -32.66*** | -28.64** | -25.86* |
| | (4.51) | (-5.01) | (-2.71) | (-2.01) |
| N | 252 | 252 | 252 | 252 |
| **R-sq** | **0.461** | **0.721** | **0.721** | **0.721** |
| **adj. R-sq** | **0.457** | **0.718** | **0.717** | **0.716** |
| **rmse** | **5.711** | **4.118** | **4.125** | **4.132** |

**t statistics in parentheses**
* p<0.05, ** p<0.01, *** p<0.001

Notice that in going from Model (1) to (2), $\bar{R}^2$ increased and the added (or *last* or *incremental*) variable (*abd*) had a t stat of 15.19, well above one in magnitude. And in going from (2) to (3), and (3) to (4), $\bar{R}^2$ decreased in both cases, and the t stats of the added variables were both less than one in magnitude.

14. So there is a direct relationship between the t stats of added/dropped variables and movements in adjusted R-squared. You should never be surprised to see this happening as you work your way through various models, adding and subtracting explanatory variables and looking at the results.
15. So if your goal is to maximize $\bar{R}^2$ (it's never a great idea to just worry about adjusted R-squared, but you wouldn't be the first analyst to do so), you want to add variables with t stats above one in magnitude and drop variables with t stats less than one in magnitude.

## Appendix

### 16. The Result:

a. The relationship between t stats and changes in adjusted R-sq:

$\bar{R}^2_{new} - \bar{R}^2_{old} > 0$ if and only if $|tstat| > 1$

### 17. The Proof:

a. Let $R^2_{old} = 1 - \frac{SSR_{old}}{SST}$ be the $R^2$ before x is added as the *last* variable in the model… and let $R^2_{new} = 1 - \frac{SSR_{new}}{SST}$ be $R^2$ with x in the model.

i. Then given the results above,

$$t^2_{\hat{\beta}_x} = (n-k-1)\frac{R^2_{new} - R^2_{old}}{1 - R^2_{new}} = (n-k-1)\frac{\left(SSR_{old} - SSR_{new}\right)/SST}{SSR_{new}/SST}$$
$$= (n-k-1)\frac{\left(SSR_{old} - SSR_{new}\right)}{SSR_{new}}.$$

ii. And so if $|tstat| > 1$ then $t^2 > 1$ and $(n-k-1)\frac{\left(SSR_{old} - SSR_{new}\right)}{SSR_{new}} > 1$, or put differently:

$(n-k-1)SSR_{old} > (n-k)SSR_{new}$.

b. But $\bar{R}^2_{new} - \bar{R}^2_{old} = \frac{n-1}{n-k}\frac{SSR_{old}}{SST} - \frac{n-1}{n-k-1}\frac{SSR_{new}}{SST}$

$$= \frac{(n-1)}{(n-k)(n-k-1)}\left[(n-k-1)SSR_{old} - (n-k)SSR_{new}\right].$$

i. And since $\frac{(n-1)}{(n-k)(n-k-1)} > 0$, the sign of $\bar{R}^2_{new} - \bar{R}^2_{old}$ matches the sign of $\left[(n-k-1)SSR_{old} - (n-k)SSR_{new}\right]$.

c. And so $\bar{R}^2_{new} - \bar{R}^2_{old} > 0$ if and only if $\left[(n-k-1)SSR_{old} - (n-k)SSR_{new}\right] > 0$ if and only if $|tstat| > 1$